

Moufang symmetry VII. Moufang transformations

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Abstract

Concept of a birepresentation for the Moufang loops is elaborated.

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1 Introduction

Groups are often said to be an algebraic abstraction of the notion of symmetry. As a slight generalization of this one can introduce the notion of the Moufang symmetry. The latter can be defined as a hypothetic kind of symmetry associated with the Moufang loops. By paraphrasing these words, we can also say that the Moufang loops are an algebraic abstraction of the Moufang symmetry.

By introducing such a notion of symmetry, one finds oneself confronted with a question about its real meaning. If one feels like looking at world affairs from viewpoint of the Moufang symmetry, one needs a suitable mathematical machinery for identification of this symmetry. As in case of groups, one really has to elaborate representation theory of the Moufang loops, and this is the logical way to get an answer to the question.

In the present paper we elaborate a concept of a *birepresentation* for the Moufang loops. Throughout the paper, ideas presented in [3] are very useful.

2 Moufang loops

A *Moufang loop* [4] (see also [2, 1, 7]) is a set G with a binary operation (multiplication) $\cdot : G \times G \rightarrow G$, denoted also by juxtaposition, so that the following three axioms are satisfied:

- 1) in equation $gh = k$, the knowledge of any two of $g, h, k \in G$ specifies the third one *uniquely*,
- 2) there is a distinguished element $e \in G$ with the property $eg = ge = g$ for all $g \in G$,
- 3) the *Moufang identity*

$$(gh)(kg) = g(hk)g \tag{2.1}$$

hold in G .

Recall that a set with a binary operation is called a *groupoid*. A groupoid G with axiom 1) is called a *quasigroup*. If axioms 1) and 2) are satisfied, the groupoid (quasigroup) G is called a *loop*. The element e in axiom 2) is called the *unit* (element) of the (Moufang) loop G .

In a (Moufang) loop, multiplication need not be neither associative nor commutative. Associative (Moufang) loops are well known and called *groups*. The *associativity* and *commutativity* laws read, respectively,

$$g \cdot hk = gh \cdot k, \quad gh = hg, \quad \forall g, h, k \in G$$

The most familiar kind of loops are those with the *associative* law, and these are called *groups*. A (Moufang) loop G is called *commutative* if the commutativity law holds in G , and (only) the commutative associative (Moufang) loops are said to be *Abelian*.

The most remarkable property of the Moufang loops is their *diassociativity*: in a Moufang loop G every two elements generate an associative subloop (group) [4]. In particular, from this it follows that

$$g \cdot gh = g^2h, \quad hg \cdot g = hg^2, \quad gh \cdot g = g \cdot hg, \quad \forall g, h \in G \quad (2.2)$$

The first and second identities in (2.2) are called the left and right *alternativity*, respectively, and the third one is said to be *flexibility*.

The unique solution of equation $xg = e$ ($gx = e$) is called the left (right) *inverse* element of $g \in G$ and is denoted as g_R^{-1} (g_L^{-1}). It follows from diassociativity of the Moufang loop that

$$g_R^{-1} = g_L^{-1} \doteq g^{-1} \quad (2.3a)$$

$$g^{-1} \cdot gh = hg \cdot g^{-1} \quad (2.3b)$$

$$(g^{-1})^{-1} = g \quad (2.3c)$$

$$(gh)^{-1} = h^{-1}g^{-1}, \quad \forall g, h \in G \quad (2.3d)$$

3 Moufang transformations

Let \mathfrak{X} be a set and let $\mathfrak{T}(\mathfrak{X})$ denote the transformation group of X . Elements of \mathfrak{X} are called *transformations* of \mathfrak{X} . Multiplication in $\mathfrak{T}(\mathfrak{X})$ is defined as composition of transformations, and unit element of $\mathfrak{T}(\mathfrak{X})$ coincides with the identity transformation id of X .

Let G be a Moufang loop with the unit element $e \in G$ and let (S, T) denote a pair of maps $S, T : G \rightarrow \mathfrak{T}(\mathfrak{X})$.

Definition 3.1 (birepresentation). The pair (S, T) is said to be an *action* of G on \mathfrak{X} if

$$S_e = T_e = \text{id} \quad (3.1a)$$

$$S_g T_g S_h = S_{gh} T_g \quad (3.1b)$$

$$S_g T_g T_h = T_{hg} S_g \quad (3.1c)$$

hold for all g, h in G . The pair (S, T) is called also a *birepresentation* of G (in $\mathfrak{T}(\mathfrak{X})$). Transformations $S_g, T_g \in \mathfrak{T}(\mathfrak{X})$ ($g \in G$) are called G -transformations or the *Moufang transformations* of \mathfrak{X} . The set of all Moufang transformations is denoted as $\mathfrak{E}_G(S, T)$.

Example 3.2. Define the left (L) and right (R) translations of G by $gh = L_g h = R_h g$. Then it follows from the (2.1) that the pair (L, R) of maps $L_g, R_g : G \rightarrow \mathfrak{T}(G)$ is a birepresentation of G in $\mathfrak{T}(G)$.

The Moufang transformations need not close, but generate a subgroup of $\mathfrak{T}(\mathfrak{X})$. This subgroup is called an *enveloping group* of $\mathfrak{E}_G(S, T)$ and is denoted as $\overline{\mathfrak{E}_G(S, T)}$. In other words, the Moufang transformations are generators of group $\overline{\mathfrak{E}_G(S, T)}$ – the enveloping group of birepresentation (S, T) of G . The defining relations of $\mathfrak{E}_G(S, T)$ are (3.1a–c). The enveloping group $\overline{\mathfrak{E}_G(S, T)}$ can be called the *multiplication group* of the birepresentation (S, T) as well.

Definition 3.3 (kernel). The set

$$K \doteq \text{Ker}(S, T) \doteq \{g \in G \mid S_g = T_g = \text{id}\}$$

is called the *kernel* of birepresentation (S, T) . If $K = \{e\}$, then birepresentation (S, T) is called *faithful* and action of the Moufang loop G on \mathfrak{X} is called *effective*.

Example 3.4. Birepresentation (L, R) is exact.

4 Properties of Moufang transformations

Proposition 4.1. *We have*

$$S_g T_g = T_g S_g, \quad \forall g \in G \quad (4.1)$$

Proof. Set $h = u$ in (3.1c) □

Proposition 4.2. *We have*

$$S_{g^{-1}} = S_g^{-1}, \quad T_{g^{-1}} = T_g^{-1}, \quad \forall g \in G \quad (4.2)$$

Proof. In (3.1b,c) first set $h = g^{-1}$:

$$S_g S_{g^{-1}} = T_g T_{g^{-1}} = \text{id}$$

Analogously, setting in (3.1b,c) $g = h^{-1}$, we have

$$S_{h^{-1}} S_h = T_{h^{-1}} T_h = \text{id}$$

Thus

$$S_g S_{g^{-1}} = S_{g^{-1}} S_g = \text{id}, \quad T_g T_{g^{-1}} = T_{g^{-1}} T_g = \text{id}, \quad \forall g \in G \quad \square$$

Lemma 4.3. *The defining relations of the Moufang transformations can equivalently be written as follows:*

$$S_e = T_e = \text{id} \quad (4.3a)$$

$$S_h T_g S_g = T_g S_{hg} \quad (4.3b)$$

$$T_h T_g S_g = S_g T_{gh} \quad (4.3c)$$

for all g, h in G .

Proof. It follows from (3.1b) and (4.2) that

$$S_h^{-1} T_g^{-1} S_g^{-1} = T_g^{-1} S_{gh}^{-1}$$

which implies

$$S_{h^{-1}} T_{g^{-1}} S_{g^{-1}} = T_{g^{-1}} S_{(gh)^{-1}} = T_{g^{-1}} S_{h^{-1} g^{-1}}$$

Thus, replacing $g^{-1} \rightarrow g$ and $h^{-1} \rightarrow h$ we obtain (4.3b). Analogously (4.3c) can be checked. □

Theorem 4.4. *The Moufang transformations satisfy the following relation:*

$$S_g S_h T_h T_g = T_h T_g S_g S_h, \quad \forall g, h \in G \quad (4.4)$$

Proof. In (3.1b) interchange g and h to obtain

$$S_h T_g S_g = T_g S_{hg}$$

and comparing the resulting formula with (3.1b) we get

$$S_g T_g S_h T_g^{-1} = T_h^{-1} S_g T_h S_h$$

which implies the the desired relation. □

Lemma 4.5. *The defining relations of the Moufang transformations satisfy the following relations:*

$$\begin{aligned} S_{g^{-1}h} &= T_g^{-1} S_g^{-1} S_h T_g \\ T_{g^{-1}h} &= S_g T_h T_g^{-1} S_g^{-1} \\ S_{hg^{-1}} &= T_g^{-1} S_g^{-1} S_g^{-1} T_g^{-1} \\ S_{hg^{-1}} &= S_g^{-1} T_g^{-1} T_h S_g \end{aligned}$$

for all g, h in G .

Theorem 4.6. *We have:*

- 1) $\text{Ker}(S, T)$ is a subloop of the Moufang loop G ,
- 2) $S_g = S_h$ ($T_g = T_h$) iff and only if $S_{g^{-1}h} = \text{id}$ ($T_{g^{-1}h} = \text{id}$),
- 3) birepresentation (S, T) is faithful if and only if from $S_g = S_h$ and $T_g = T_h$ follows $g = h$.

Proof. Use Lemma 4.3. □

5 Triality

Define the *quadratic* Moufang transformations as

$$P_g \doteq S_g^{-1} T_g^{-1} \in \overline{\mathfrak{E}_G(S, T)}, \quad g \in G \quad (5.1)$$

Note that P_g commutes both with S_g and T_g . Thus we can equivalently define P_g by the symmetric relation

$$S_g T_g P_g \doteq \text{id}, \quad g \in G \quad (5.2)$$

Proposition 5.1. *We have*

$$\begin{aligned} P_e &= \text{id} \\ P_g^{-1} P_g &= P_g P_g^{-1} = \text{id}, \quad \forall g \in G \end{aligned}$$

Corollary 5.2. *We have*

$$P_g^{-1} = P_{g^{-1}}, \quad \forall g \in G$$

Denote by (S, T, P) the triple of maps $S, T, P : g \rightarrow \mathfrak{T}(\mathfrak{X})$.

Theorem 5.3. *Let (S, T) be a birepresentation of the Moufang loop G . Then the following pairs are birepresentations of G as well:*

$$\begin{aligned} (T^{-1}, S^{-1}) &: g \rightarrow T_g^{-1}, \quad S_g^{-1} \\ (T, P) &: g \rightarrow T_g, \quad g \rightarrow P_g \\ (P^{-1}, P^{-1}) &: g \rightarrow P_g^{-1}, \quad T_g^{-1} \\ (P, S) &: g \rightarrow P_g, \quad g \rightarrow S_g \\ (S^{-1}, P^{-1}) &: g \rightarrow S_g^{-1}, \quad P_g^{-1} \end{aligned}$$

Proof. As an example, check the definig relations for pair (T, P) :

$$\begin{aligned} T_g P_g T_h &= T_{gh} P_g \\ T_g P_g P_{gh} &= P_{hg} T_g \end{aligned}$$

To get the first relation, express P_g from (5.2) and replace into this relations, the resulting relation is equivalent to (3.1b). The second relation can equivalently be written as

$$P_{hg^{-1}} = S_g P_h T_g$$

Now calculate:

$$\begin{aligned} P_{hg^{-1}}(5.1) &= S_{hg^{-1}}^{-1} T_{hg^{-1}}^{-1} \\ (4.2), (2.3d) &= S_{gh^{-1}} T_{gh^{-1}} \\ (3.1b), (4.3c), (4.2) &= S_g T_g S_h^{-1} T_g^{-1} S_g^{-1} T_h^{-1} T_g S_g \\ (4.1) &= S_g T_g S_h^{-1} S_g^{-1} T_g^{-1} T_h^{-1} T_g S_g \\ (4.4) &= S_g T_g T_g^{-1} T_h^{-1} S_h^{-1} S_g^{-1} T_g S_g \\ (5.1) &= S_g T_h T_g \end{aligned}$$

The defining relations for other pairs can be checked analogously. \square

Corollary 5.4. *The defining relations of the birepresentatitons from triple (S, T, P) can be collected to the following table:*

(S, T)	(T^{-1}, S^{-1})	(T, P)	(P^{-1}, T^{-1})	(P, S)	(S^{-1}, P^{-1})
(5.3a)	(5.3b)	(5.3b)	(5.3c)	(5.3c)	(5.3a)
(5.4b)	(5.4a)	(5.4c)	(5.4b)	(5.4a)	(5.4c)

where

$$S_{g^{-1}h} \stackrel{(a)}{=} P_x S_h T_g, \quad T_{g^{-1}h} \stackrel{(b)}{=} S_x T_h P_g, \quad P_{g^{-1}h} \stackrel{(c)}{=} T_x P_h S_g \quad (5.3)$$

$$S_{hg^{-1}} \stackrel{(a)}{=} T_g S_h P_g, \quad T_{hg^{-1}} \stackrel{(b)}{=} P_g T_h S_g, \quad P_{hg^{-1}} \stackrel{(c)}{=} S_g P_h T_g \quad (5.4)$$

Corollary 5.5. *It follows from (5.3a-c) and (5.4a-c) that*

$$P_x S_h^{-1} T_g = T_g S_h^{-1} P_g, \quad S_x T_h^{-1} P_g = P_g T_h^{-1} S_g, \quad T_x P_h^{-1} S_g = S_g P_h^{-1} T_g$$

The latter are equivalent to (4.1) and to

$$T_g T_h P_h P_g = P_h P_g T_g T_h, \quad T_g T_h P_h P_g = P_h P_g T_g T_h$$

Collecting above properties of birepresentations we can propose

Theorem 5.6 (principle of triality). *The definig relations of the Moufang transformations are invariant under the triality substitutions*

$$\begin{aligned} \text{id} &= (S \rightarrow S)(T \rightarrow T)(P \rightarrow P) \\ \tau &= (S \rightarrow T^{-1} \rightarrow S)(P \rightarrow P^{-1}) \\ \rho &= (S \rightarrow T \rightarrow P \rightarrow S) \\ \rho^2 &= (S \rightarrow P \rightarrow T \rightarrow S) \\ \rho \circ \tau &= (S \rightarrow P^{-1} \rightarrow S)(T \rightarrow T^{-1}) \\ \rho^2 \circ \tau &= (T \rightarrow P^{-1} \rightarrow P)(S \rightarrow S^{-1}) \end{aligned}$$

Hence all algebraic consequences of the defining relations must be triality invariant as well.

6 Reconstruction Theorem

It turns out that the triality symmetry is a characteristic property of the Moufang transformations.

Theorem 6.1 (reconstruction). *Let G be a groupoid and (S, TP) a triple of maps $S, T, P : G \rightarrow \mathfrak{T}(\mathfrak{X})$ such that:*

- 1) $S_g T_g P_g = \text{id}$ for all g in G ,
- 2) for every g in G there exists \bar{g} in G such that $S_x^{-1} = S_{\bar{g}}$ and $T_x^{-1} = T_{\bar{g}}$,
- 3) for all g, h in G relations

$$\begin{aligned} S_{\bar{g}h} &= P_x S_h T_g, & T_{\bar{g}h} &= S_x T_h P_g, & P_{\bar{g}h} &= T_x P_h S_g \\ S_{h\bar{g}} &= T_g S_h P_g, & T_{h\bar{g}} &= P_g T_h S_g, & P_{h\bar{g}} &= S_g P_h T_g \end{aligned}$$

are satisfied in $\mathfrak{T}(\mathfrak{X})$,

- 4) from $S_g = S_h$ and $T_g = T_h$ it follows that $g = h$.

Then G is a Moufang loop. The unit element of G is $g\bar{g} = \bar{g}g \doteq e$, where the latter does not depend on the choice of g in G , and the inverse element of g is \bar{g} .

Proof. The detailed proof is presented in [6] □

7 Triple closure

Theorem 7.1. *The Moufang transformations satisfy the triple closure relations:*

$$S_g S_h S_g \stackrel{(a)}{=} S_{ghg}, \quad T_g T_h T_g \stackrel{(b)}{=} T_{ghg}, \quad P_g P_h P_g \stackrel{(c)}{=} P_{ghg}, \quad \forall g \in G \quad (7.1)$$

Proof. Calculate:

$$\begin{aligned} S_{ghg} &\stackrel{(5.3)}{=} P_{g^{-1}} S_{hg} T_{g^{-1}} \\ &\stackrel{(5.4)}{=} P_{g^{-1}} T_{g^{-1}} S_h P_{g^{-1}} T_{g^{-1}} \\ &\stackrel{(5.2)}{=} S_{g^{-1}}^{-1} S_h S_{g^{-1}}^{-1} \\ &\stackrel{(4.2)}{=} S_g S_h S_g \end{aligned}$$

The remaining relations (7.1b,c) can be checked analogously. □

Remark 7.2. It follows from Theorem (7.1) that the Moufang transformations realize the *triple family of transformations* [5] of \mathfrak{X}

8 Minimality conditions

We call birepresentation (S, T) *associative* if the Moufang transformations satisfy the closure relations

$$S_g S_h \stackrel{(a)}{=} S_{gh}, \quad T_g T_h \stackrel{(b)}{=} T_{gh}, \quad S_g T_h \stackrel{(c)}{=} T_h S_g, \quad \forall g, h \in G \quad (8.1)$$

It follows from (3.1b,c) that these conditions are equivalent.

It has to be noted that the non-associative Moufang loops do not have faithful associative birepresentations. Really, for the associative birepresentation we have

$$S_g S_h S_k = S_{gh \cdot k} = S_{g \cdot hk}, \quad T_g T_h T_k = S_{gh \cdot k} = T_{g \cdot hk}$$

from which it follows that $(gh \cdot k)^{-1}(g \cdot hk) \in \text{Ker}(S, T)$. But for the faithful birepresentation $\text{Ker}(S, T) = \{e\}$, hence $g \cdot hk = g \cdot hk$.

Denote the commutator of transformations A, B by $[A, B] \doteq ABA^{-1}B^{-1}$. Equivalence of the associativity constraints (8.1) can be also seen from

Theorem 8.1 (minimality conditions). *The Moufang transformations satisfy relations*

$$[T_h, S_g^{-1}] \stackrel{(a)}{=} S_g^{-1} S_g S_h \stackrel{(b)}{=} T_{gh} T_g^{-1} T_h^{-1} \stackrel{(c)}{=} [S_g^{-1}, T_h] \stackrel{(d)}{=} S_g^{-1} S_h^{-1} S_{hg} \stackrel{(e)}{=} T_g T_h T_{hg}^{-1} \quad (8.2)$$

Proof. It is easy to check that (8.2a) \equiv (5.3a), (8.2b) \equiv (5.3c), (8.2c) \equiv (5.3b), (8.2e) \equiv (5.3c) and (8.2a) \equiv (5.3a). Note that other possible equalities from (8.2a–e) give rise also (5.3a–c), (5.4a–c) or the triple closure relations (7.1a–c). \square

Definition 8.2 (associators). Let (S, T) be a birepresentation of the Moufang loop G . Elements from group $\mathfrak{E}_G(S, T)$ of form

$$\begin{aligned} S(g; h) &\doteq S_{gh}^{-1} S_g S_h \\ T(g; h) &\doteq T_{gh} T_g^{-1} T_h^{-1} \\ [T_g, S_h^{-1}] &\doteq T_g S_h^{-1} T_g^{-1} S_h \\ [S_g^{-1}, T_h] &\doteq S_g^{-1} T_h S_g T_h^{-1} \end{aligned}$$

are called *associators* of birepresentation (S, T) .

It is easy to see from Theorem 8.2:

Corollary 8.3 (minimality conditions). *Associator of a birepresentation (S, T) satisfy the minimality conditions*

$$[T_g, S_h^{-1}] = S(g; h) = T(g; h) = [S_g^{-1}, T_h] = S^{-1}(h; g) = T^{-1}(h; g) \quad (8.3)$$

Remark 8.4. For associative Moufang transformations we have

$$[T_g, S_h^{-1}] = S(g; h) = T(g; h) = [S_g^{-1}, T_h] = S^{-1}(h; g) = T^{-1}(h; g) = \text{id} \quad (8.4)$$

Comparing (8.3) and (8.4) one can say that the Moufang transformations have the property that their associativity is spoiled in the *minimal* way. Constraints (8.2) and (8.3) are hence called the *minimality conditions*

By triality we can propose

Theorem 8.5 (triality and minimality). *The Moufang transformations satisfy the minimality conditions:*

$$[P_h, T_g^{-1}] \stackrel{(a)}{=} T_{gh}^{-1} T_g T_h \stackrel{(b)}{=} P_{gh} P_g^{-1} P_h^{-1} \stackrel{(c)}{=} [T_g^{-1}, T_h] \stackrel{(d)}{=} T_g^{-1} T_h^{-1} T_{hg} \stackrel{(e)}{=} P_g P_h P_{hg}^{-1} \quad (8.5)$$

$$[S_h, P_g^{-1}] \stackrel{(a)}{=} P_{gh}^{-1} P_g P_h \stackrel{(b)}{=} S_{gh} S_g^{-1} S_h^{-1} \stackrel{(c)}{=} [P_g^{-1}, P_h] \stackrel{(d)}{=} P_g^{-1} P_h^{-1} P_{hg} \stackrel{(e)}{=} S_g S_h S_{hg}^{-1} \quad (8.6)$$

Proof. Constraints (8.5a–e) and (8.6a–e) hold because (T, P) and (P, S) are birepresentations of G . \square

9 Theorem on kernel of birepresentation

Definition 9.1 (normal divisor [2]). A subloop N of the Moufang loop G is called a *normal divisor* of G if it is invariant with respect to the following transformations of \mathfrak{X} from the group $\mathfrak{E}_G(S, T)$:

$$L(g; h) \doteq L_{gh}^{-1} L_g L_h, \quad M_g^+ \doteq R_g L_g^{-1}$$

If $L(g; h) = \text{id}$ for all g, h in G , the Moufang loop G is a group and then every M_g^+ ($g \in G$) is an *inner automorphism* of G

Theorem 9.2. *The kernel $\text{Ker}(S, T)$ of a birepresentation (S, T) of the Moufang loop G is a normal divisor of G .*

Proof. We know from Theorem 4.6 $\text{Ker}(S, T)$ is a subloop of G , thus it is sufficient to check that for all g, h in G and k in $\text{Ker}(S, T)$ we have

$$S_{M_g^+ k} \stackrel{(a)}{=} \text{id}, \quad T_{M_g^+ k} \stackrel{(a)}{=} \text{id} \tag{9.1}$$

$$S_{L(g, h)k} \stackrel{(a)}{=} \text{id}, \quad T_{L(g, h)k} \stackrel{(a)}{=} \text{id} \tag{9.2}$$

First calculate

$$\begin{aligned} S_{M_g^+ k} &= S_{R_g L_g^{-1} k} \\ &= T_g^{-1} S_{L_g^{-1} k} P_g^{-1} \\ &= T_g^{-1} P_g S_k T_g P_g^{-1} \\ &= T_g^{-1} P_g \text{id} T_g P_g^{-1} \\ &= T_g^{-1} P_g T_g P_g^{-1} \\ &= T_g^{-1} S_g^{-1} P_g^{-1} \\ &= (P_g S_g T_g)^{-1} \\ &= \text{id} \end{aligned}$$

Condition (9.1b) can be checked analogously. Next calculate

$$\begin{aligned} S_{L(g, h)k} &= S_{L_{gh}^{-1} L_g L_h k} \\ &= P_{gh} S_{L_g L_h k} T_{gh} \\ &= P_{gh} P_g^{-1} S_{L_h k} T_g^{-1} T_{gh} \\ &= P_{gh} P_g^{-1} P_h^{-1} S_k T_h^{-1} T_g^{-1} T_{gh} \\ &= P_{gh} P_g^{-1} P_h^{-1} \text{id} T_h^{-1} T_g^{-1} T_{gh} \\ &= P_{gh} P_g^{-1} P_h^{-1} T_h^{-1} T_g^{-1} T_{gh} \\ (8.5b) &= \text{id} \end{aligned}$$

Condition (9.2b) can be checked analogously. □

10 Birepresentation of quotient loop $G / \text{Ker}(S, T)$

Recall some basic facts [2] from theory of the Moufang loops.

Let N be a normal divisor of (S, T) . Then we can define on the Moufang loop G the *left* (*right*) *equivalence*: for g, h in G we set $g \stackrel{L}{\sim} h$ ($g \stackrel{R}{\sim} h$) if $g^{-1}h \in N$ ($hg^{-1} \in N$). The resulting

equivalence classes are called the *left (right) cosets* with respect to the normal divisor N . It turns out [2] that the left and right cosets can be presented as gN and Ng , respectively, and coincide: $gN = Ng$. On the set of cosets of G with respect to N we can define multiplication:

$$(gN)(hN) \doteq (gh)N$$

which satisfy all the Moufang loop axioms. The resulting Moufang loop is called the quotient loop with respect to N and is denoted by G/N . The unit element of G/N is K .

The normal divisors coincide [2] with kernels of homomorphisms.

Theorem 10.1. *Let (S, T) be a birepresentation of the Moufang loop G and $K \doteq \text{Ker}(S, T)$ be kernel of the birepresentation (S, T) . Then the pair of maps $gK \rightarrow S_g$, $gk \rightarrow T_g$ is a faithful birepresentation of the quotient Moufang loop G/K .*

Proof. A pair (S', T') of maps $gK \rightarrow S'_{gK}$, $gk \rightarrow T'_{gK}$ is a birepresentation of G if the following conditions are satisfied:

$$S'_K = T'_K = \text{id} \tag{10.1a}$$

$$S'_{gk} T'_{gK} S'_{hK} = S'_{(gK)(hK)} T'_{gK} = S'_{(gh)K} T'_{gK} \tag{10.1b}$$

$$S'_{gk} T'_{gK} T'_{hk} = T'_{(hK)(gK)} S'_{gK} = T'_{(hg)K} S'_{gK} \tag{10.1c}$$

Define S'_{gK} and S'_{gK} by the following simple formulae:

$$S'_{gK} \doteq S_g, \quad T'_{gK} \doteq T_g, \quad \forall gK \in G/K$$

First of all, note that the definition of S'_{gK} and S'_{gK} does depend on the choice of representatives in coset gK . Really, if $k \in gK$, then $k = gn$, where $n \in K$. Then we have

$$\begin{aligned} S'_{kK} &= S_k = S_{gn} = P_g^{-1} S_n T_g^{-1} = P_g^{-1} T_g^{-1} = S_g \\ T'_{kK} &= T_k = T_{gn} = T_g^{-1} S_n P_g^{-1} = T_g^{-1} P_g^{-1} = T_g \end{aligned}$$

Thus the maps $gK \rightarrow S'_{gK}$, $gk \rightarrow T'_{gK}$ are defined uniquely. The definig relations of (10.1b,c) follow from (3.1a–c) and the above definition of S'_{gK} and $'S_{gK}$. This means that the pair of maps $gK \rightarrow S'_{gK}$, $gk \rightarrow T'_{gK}$ is a birepresentation of G/K . The set

$$\text{Ker}(S', T') \doteq \{gK \in G/K \mid S'_{gK} = T'_{gK} = \text{id}\}$$

is the kernel of birepresentation (S', T') . Evidently, $K \in \text{Ker}(S', T')$. If $gK \in \text{Ker}(S', T')$, then it follws from

$$S'_{gK} = S_g = \text{id}, \quad T'_{gK} = T_g = \text{id}$$

that $g \in K$. Thus, $\text{Ker}(S', T') = \{K\}$, from which it follows that birepresentation $gK \rightarrow S'_{gK} \doteq S_g$, $gk \rightarrow T'_{gK} \doteq T_g$ is faithful. \square

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